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THE STABILITY DOMAINS OF HAMILTONIAN SYSTEMS†

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Linear Hamiltonian systems with an arbitrary number of degrees of freedom, which depend smoothly on a vector of real parameters, are investigated. All possible singularities of the boundary of the stability domain of Hamiltonian systems of general position are determined and described for the case of two and three parameters. In the first approximation, the geometry of these singularities (the orientation in the parameter space, angles, etc.) is determined on the basis of the first derivative of the matrix of the system with respect to the parameters, as are the eigenvectors and generalized eigenvectors evaluated at the singular point. A detailed investigation is made of gyroscopic systems as a special case of Hamiltonian systems. As mechanical examples, an account is given of the problem of the stability of the oscillations of a tube through which a fluid is flowing, and of the stability of the motion of a two-body system. The tangent cones to the stability domains of these systems at singular points of the "cusp" and "dihedral angle" type, which arise on the boundaries of these domains, are found. © 1999 Elsevier Science Ltd. All rights reserved.

All types of Jordan structures that may occur in the general position for two- and three-parameter families of Hamiltonian matrices have been listed [1]. In what follows we will determine and describe all forms of singularity of the boundary of the stability domain in the general position for Hamiltonian and gyroscopic systems in the case of two and three parameters. The geometry of these singularities will be determined in the first approximation from the first derivatives of the matrices of the system with respect to the parameters, as well as the eigenvectors and generalized eigenvectors evaluated at the singularity. The proofs are based on perturbation theory for eigenvalues of matrices that depend on parameters [2, 3], and on the theory of miniversal deformations of Hamiltonian matrices [1]. The methods developed below may be used to investigate singularities in the case of more than three parameters. These methods and the results obtained through their application constitute a further development and extension of the approach to investigating singularities used previously [4] to study singularities of the boundaries of stability domains of families of non-symmetric matrices.

1. BIFURCATIONS OF THE IMAGINARY EIGENVALUES OF HAMILTONIAN MATRICES

We will consider a mechanical system with m degrees of freedom described by canonical Hamiltonian variables $q_1, q_2, \ldots, q_m, p_1, p_2, \ldots, p_m$. We will represent these variables as the components of a vector $\mathbf{x} \in \mathbb{R}^{2m}$, $q_i = x_i$, $p_i = x_{m+i}$ $(i = 1, \ldots, m)$ and assume that the mechanical system has a Hamiltonian $H = \mathbf{x}^T\mathbf{A}\mathbf{x}/2$, where A is a real symmetric square matrix of order $2m$ which depends smoothly on a vector of real parameters $\mathbf{h} = (h_1, h_2, \ldots, h_n)^T$ but does not depend on time.

If we introduce a square partitioned matrix

$$\mathbf{J} = \begin{Vmatrix} \mathbf{O} & \mathbf{I}_m \\ -\mathbf{I}_m & \mathbf{O} \end{Vmatrix}$$

where $\mathbf{I}_m$ and $\mathbf{O}$ are the square identity and zero matrices of order m, respectively, then the system of canonical Hamilton equations $\dot{\mathbf{q}} = H_p$, $\dot{\mathbf{p}} = -H_q$ may be written in the form [5]

$$d\mathbf{x}/dt = \mathbf{J}\mathbf{A}\mathbf{x} \tag{1.1}$$

Systems described by Eqs (1.1) are known as linear Hamiltonian systems and the matrix $\mathbf{JA}$ is called a Hamiltonian matrix.

Consider the eigenvalue problem for system (1.1)

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$$[\mathbf{JA} - \lambda\mathbf{I}]\mathbf{u} = 0$$

where λ is an eigenvalue $\mathbf{u}$ is an eigenvector of dimension $2m$ and $\mathbf{I}$ is the identity matrix of order $2m$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2m}$ are determined from the characteristic equation $[\mathbf{JA} - \lambda\mathbf{I}] = 0$. The complex eigenvalues of the Hamiltonian matrix $\mathbf{JA}$ form quadruplets λ, $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$ of points symmetrically placed with respect to the imaginary and real axes in the complex plane, and doublets λ, $-\lambda$ when they are real or pure imaginary; the algebraic multiplicity of the eigenvalue $\lambda = 0$ must be an even number [6].

Consider a point $\mathbf{h} = \mathbf{h}_0$ in the parameter space. For our subsequent investigations we will need relations that define the bifurcation of multiple eigenvalues of the matrix $\mathbf{JA}_0 = \mathbf{JA}(\mathbf{h}_0)$ when the parameters are perturbed. To that end, we impart an increment to the parameter vector: $\mathbf{h} = \mathbf{h}_0 + \varepsilon\mathbf{e}$, where $\varepsilon > 0$ is a small parameter and $\mathbf{e} \in \mathbb{R}^n$ an arbitrary variation vector. Because of the perturbation in the parameter vector, the eigenvalues will also receive increments which, depending on the Jordan structure of the matrix $\mathbf{JA}_0$, admit of different representations.

1. Suppose the matrix $\mathbf{JA}_0$ has a pure imaginary eigenvalue $\lambda_0 = i\omega$ (for a zero eigenvalue, $\omega = 0$), associated with which is a Jordan cell of order k. Let $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ denote the Jordan chains of the direct and adjoint problems, which satisfy the following equations

$$\begin{aligned}
&[\mathbf{JA}_0 - i\omega\mathbf{I}]\mathbf{u}_0 = 0, && [\mathbf{JA}_0 - i\omega\mathbf{I}]^*\mathbf{v}_0 = 0\\
&[\mathbf{JA}_0 - i\omega\mathbf{I}]\mathbf{u}_1 = \mathbf{u}_0, && [\mathbf{JA}_0 - i\omega\mathbf{I}]^*\mathbf{v}_1 = \mathbf{v}_0\\
&\quad\vdots && \quad\vdots\\
&[\mathbf{JA}_0 - i\omega\mathbf{I}]\mathbf{u}_{k-1} = \mathbf{u}_{k-2}, && [\mathbf{JA}_0 - i\omega\mathbf{I}]^*\mathbf{v}_{k-1} = \mathbf{v}_{k-2}
\end{aligned}$$

We introduce normalization conditions $\mathbf{v}_0^*\mathbf{u}_{k-1} = 1$, $\mathbf{v}_j^*\mathbf{u}_{k-1} = 0$ $(j = 1, \ldots, k-1)$, which, given fixed $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$, uniquely determine the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$. Define vectors $\mathbf{f}_j = (f_j^1, f_j^2, \ldots, f_j^n)^T \in \mathbb{R}^n$ $(j = 0, \ldots, k-1)$ with components

$$f_j^l = -i^{k-j} \sum_{r=0}^{j} \mathbf{v}_r^* \mathbf{J} \frac{\partial \mathbf{A}}{\partial h_l} \mathbf{u}_{j-r}, \quad l = 1, \ldots, n \tag{1.2}$$

where the derivatives with respect to the parameters are evaluated at the point $\mathbf{h} = \mathbf{h}_0$. The vectors $\mathbf{f}_0, \ldots, \mathbf{f}_{k-1}$ are real and do not depend on the choice of the chain of vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$. They will be used later to describe the geometry of the singularities.

The case $k = 2$ is considered separately. Here, the expressions for the components of the vector $\mathbf{f}_0$ may be transformed to

$$f^s = -\left(\mathbf{u}_0^* \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}_0\right)(\mathbf{u}_1^*[\mathbf{A}_0 + i\omega\mathbf{J}]\mathbf{u}_1)^{-1} \tag{1.3}$$

For convenience, the zero subscript of the vector $\mathbf{f}_0$ is omitted in (1.3) and in what follows (for $k = 2$). In terms of the vector $\mathbf{f}$, the bifurcation of a double eigenvalue $\lambda_0 = i\omega$ into two simple eigenvalues is described as follows [2, 3]:

$$\lambda = i\omega \pm \sqrt{(\mathbf{f}, \mathbf{e})\varepsilon} + O(\varepsilon) \tag{1.4}$$

(the parentheses denote the scalar product in $\mathbf{R}^n$). It follows from the symmetry of the eigenvalues about the imaginary axis and from (1.4) that a double eigenvalue $\lambda_0 = i\omega$ bifurcates into two pure imaginary eigenvalues if $(\mathbf{f}, \mathbf{e}) < 0$.

2. Consider the case in which the matrix $\mathbf{JA}_0$ has a double imaginary eigenvalues $\lambda_0 = i\omega \neq 0$, associated with which are two linearly independent eigenvectors $\mathbf{u}'$ and $\mathbf{u}''$. We choose these vectors so that they satisfy an orthogonality condition $\mathbf{u}'^*\mathbf{J}\mathbf{u}'' = 0$. Define real constants b_1, b_2 and vectors $g_j = (g_j^1, \ldots, g_j^n)^T \in \mathbb{R}^n$ $(j = 1, 2, 3)$ by the formulae

$$\begin{gathered}
b_1 = -i\mathbf{u}'^*\mathbf{J}\mathbf{u}', \quad b_2 = -i\mathbf{u}''^*\mathbf{J}\mathbf{u}''\\
g_1^s = \left(\mathbf{u}'^* \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}'\right) b_2 - \left(\mathbf{u}''^* \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}''\right) b_1, \quad g_2^s + i g_3^s = 2\sqrt{|b_1 b_2|}\left(\mathbf{u}'^* \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}''\right)
\end{gathered} \tag{1.5}$$

where the derivatives with respect to the parameters are evaluated at the point $\mathbf{h} = \mathbf{h}_0$.

The bifurcation of the double eigenvalues $\lambda_0 = i\omega$ when the parameters are perturbed is determined by the expression $\lambda = i\omega + \varepsilon\mu + o(\varepsilon)$, where the two values of the first correction μ are found by solving the following quadratic equation [2, 3]

$$\det[\mathbf{u}_i^* \mathbf{A}_1 \mathbf{u}_j + \mu \mathbf{u}_i^* \mathbf{J} \mathbf{u}_j] = 0, \quad i, j = 1, 2; \quad \mathbf{A}_1 = \sum_{k=1}^{n} \frac{\partial \mathbf{A}}{\partial p_k} e_k$$

It can be shown that the double eigenvalue λ_0 bifurcates into two pure imaginary simple eigenvalues if

$$D = -(\mathbf{g}_1, \mathbf{e})^2 - \operatorname{sign}(b_1 b_2)[(\mathbf{g}_2, \mathbf{e})^2 + (\mathbf{g}_3, \mathbf{e})^2] < 0 \tag{1.6}$$

and into two complex eigenvalues with non-zero real parts of opposite sign if $D > 0$.

3. Consider the case in which the matrix $\mathbf{JA}_0$ has a double eigenvalue λ_0 with linearly independent eigenvectors $\mathbf{u}'$ and $\mathbf{u}''$ which we choose to be real and which satisfy the normalization conditions $\mathbf{u}'^T \mathbf{J} \mathbf{u}'' = 1$. We introduce $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_{12} \in \mathbb{R}^n$ with components

$$k_1^s = \mathbf{u}'^T \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}', \quad k_2^s = \mathbf{u}''^T \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}'', \quad k_{12}^s = \mathbf{u}'^T \frac{\partial \mathbf{A}}{\partial h_s} \mathbf{u}'', \quad s = 1, \ldots, n \tag{1.7}$$

In terms of these vectors, the bifurcation of the double eigenvalue $\lambda_0 = 0$ into two simple eigenvalues is described by the relation

$$\lambda = \pm\sqrt{D_1}\,\varepsilon + o(\varepsilon), \quad D_1 = (\mathbf{k}_{12}, \mathbf{e})^2 - (\mathbf{k}_1, \mathbf{e})(\mathbf{k}_2, \mathbf{e}) \tag{1.8}$$

If $D_1 < 0$, the double zero eigenvalue bifurcates into two pure imaginary simple eigenvalues.

2. SINGULARITIES OF THE BOUNDARIES OF STABILITY DOMAINS OF FAMILIES OF HAMILTONIAN MATRICES

Let us investigate the stability of the trivial solution $\mathbf{x} = 0$ of the Hamiltonian system (1.1). The solution is stable in Lyapunov's sense if and only if all the eigenvalues are pure imaginary and semi-simple. The point $\mathbf{h} = \mathbf{h}_0$, to which there correspond only simple pure imaginary eigenvalues $\lambda = \pm i\omega \neq 0$, is always an interior point of the stability domain, while the points on the boundary of the stability domain (BSD) are characterized by the existence of multiple pure imaginary or zero λ (when the other eigenvalues are simple and pure imaginary) [6].

We shall classify the points of the BSD according to the type of Jordan structure of the Hamiltonian matrix, which will be denoted by the product of the determinants of the Jordan cells corresponding to multiple eigenvalues [7]. For example, $(\pm i\omega_1)^3(\pm i\omega_2)^2$ means that there are a pair of three-fold eigenvalues $\lambda = \pm i\omega_1 \neq 0$ with Jordan cells of order 3 and a pair of double eigenvalues $\lambda = \pm i\omega_2 \neq 0$, $\omega_2 \neq \omega_1$ with Jordan cells of order 2; 00 denotes the existence of a double eigenvalue $\lambda = 0$ with two Jordan cells of first order; and so on. It is understood that unwritten eigenvalues are simple and pure imaginary.

The BSD consists of smooth hypersurfaces, which may have various singularities. At non-singular points they are described by matrices of types 0^2 and $(\pm i\omega)^2$ [1]. The bifurcation of a double eigenvalue $\lambda_0 = i\omega$ (or $\lambda_0 = 0$) in the neighbourhood of a non-singular point of the BSD is described by (1.4). Note that the other eigenvalues remain simple and pure imaginary in the neighbourhood of that point [6]. It follows from (1.4) that when $(\mathbf{f}, \mathbf{e}) < 0$ the double eigenvalue λ_0 splits along the imaginary axis (stability). But when $(\mathbf{f}, \mathbf{e}) > 0$, perturbation of the parameters results in the appearance of an eigenvalue with positive real part (instability). Vectors $\mathbf{e}$ tangent to the BSD are determined from the condition $(\mathbf{f}, \mathbf{e}) = 0$. Consequently, the vector $\mathbf{f}$, evaluated at the point under consideration for $\lambda_0 = i\omega$ (or $\lambda_0 = 0$), is normal to the BSD and lies in the instability domain. The static and oscillatory forms of stability loss at the points 0^2 and $(\pm i\omega)^2$ are known in the engineering literature as divergence and flutter, respectively.

In the case of general position, the types of singular points of the BSD of two-parameter families of Hamiltonian matrices are singled out from the general list of singularities of such matrices, which has been determined before [1]: 0^4, $(\pm i\omega)^3$, $0^2(\pm i\omega)^2$, $(\pm i\omega)^2\,(\pm i\omega_2)^2$.

In the sequel we will need the concept of the tangent cone [8]. The tangent cone to the stability domain at one of its boundary points is the set of directions of vectors $\mathbf{e}$ along which there is a curve, lying within the stability domain, issuing from the point in question. The tangent cone is said to be non-degenerate if it cuts out a set of non-zero measure on a sphere. The tangent cone describes the stability domain in the neighbourhood of the point in the linear approximation and contains the basic information about the geometry of the singularity. In this section we shall consider singularities with non-degenerate tangent cones; singular points with degenerate tangent cones will be considered in the next section.

Consider a point of type $0^2(\pm i\omega)^2$. By analogy with the points of type 0^2 and $(\pm i\omega)^2$ considered previously, if $(\mathbf{f}^0, \mathbf{e}) < 0$, $(\mathbf{f}^{i\omega}, \mathbf{e}) < 0$ (the subscript of the vector denotes the eigenvalue for which it is evaluated), then the double eigenvalues $\lambda = 0$ and $\lambda = \pm i\omega$ bifurcate along the imaginary axis (stability). If one of the inequalities is reversed, perturbation induces the appearance of an eigenvalue such that $\mathrm{Re}\lambda > 0$ (instability). The singular point in question is a corner point of the BSD. The vectors $\mathbf{f}^0$ and $\mathbf{f}^{i\omega}$ are the normals, directed into the stability domain, to the corresponding sides of the angle (Fig. 1, the stability domain is denoted by S). The singularity $(\pm i\omega_1)^2(\pm i\omega_2)^2$ is also a corner point of the BSD, for which the vectors $\mathbf{f}^{i\omega 1}$ and $\mathbf{f}^{i\omega 2}$ are, similarly, normals to the sides of the angle, directed into the stability domain (Fig. 1). The tangent cones to the stability domain at points of type $0^2(\pm i\omega)^2$ and $(\pm i\omega_1)^2(\pm i\omega_2)^2$ have the form

$$K_{0^2(\pm i\omega)^2} = \{\mathbf{e} : (\mathbf{f}^0, \mathbf{e}) \leqslant 0, \quad (\mathbf{f}^{i\omega}, \mathbf{e}) \leqslant 0\} \tag{2.1}$$

$$K_{(\pm i\omega_1)^2(\pm i\omega_2)^2} = \{\mathbf{e} : (\mathbf{f}^{i\omega_1}, \mathbf{e}) \leqslant 0, \quad (\mathbf{f}^{i\omega_2}, \mathbf{e}) \leqslant 0\} \tag{2.2}$$

(the inequalities are not strict because a tangent cone is a closed set). The remaining singularities of two-parameter families 0^4 and $(\pm i\omega)^3$ are cuspidal points (degenerate tangent cones) and will be considered in the next section.

With three parameters, the singularities of the BSD in general position comprise smooth curves of the type 0^4, $(\pm i\omega)^3$, $0^2(\pm i\omega)^2$, $(\pm i\omega_1)^2(\pm i\omega_2)^2$, as well as isolated points of the types $(\pm i\omega)(\pm i\omega)$, 00, $(\pm i\omega)^4$, 0^6, $0^4(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$ [1].

Curves of the type $^2(\pm i\omega_1{}^2)$ and $(\pm i\omega_1)^2(\pm i\omega_2)^2$ form an edge of the BSD, as in Fig. 2. The singularity at a point of such a curve is called a "dihedral angle", and the tangent cone is described by Eqs (2.1) and (2.2). The vectors $\mathbf{f}^0$, $\mathbf{f}^{i\omega}$ for a point of an edge of type $0^2(\pm i\omega)^2$ are normals to the sides of the dihedral angle, lying in the instability domain. The tangent vector to the edge e_τ is orthogonal to both vectors $\mathbf{f}^0$ and $\mathbf{f}^{i\omega}$ and may be found as their vector product $\mathbf{e}_\tau = \mathbf{f}^0 \times \mathbf{f}^{i\omega}$. Similarly, the tangent vector to an edge of type $(\pm i\omega_1)^2(\pm i\omega_2)^2$ may be found as $\mathbf{e}_\tau = \mathbf{f}^{i\omega_1} \times \mathbf{f}^{i\omega_2}$.

At points $0^2(\pm i\omega_1)^2(\pm i\omega_2)^2$ and $(\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2$ one obtains singularities of the "trihedral angle" type (Fig. 2). These points differ from $0^2(\pm i\omega)^2$ and $(\pm i\omega_1)^2(\pm i\omega_2)^2$ in the presence of one more pair of double eigenvalues of the type $(\pm i\omega)^2$, which leads to another restriction on the vector $\mathbf{e}$ lying in the stability domain. The tangent cones to the stability domain are defined by the relations

$$K_{0^2(\pm i\omega_1)^2(\pm i\omega_2)^2} = \{\mathbf{e} : (\mathbf{f}^0, \mathbf{e}) \leqslant 0, (\mathbf{f}^{i\omega_1}, \mathbf{e}) \leqslant 0, (\mathbf{f}^{i\omega_2}, \mathbf{e}) \leqslant 0\} \tag{2.3}$$

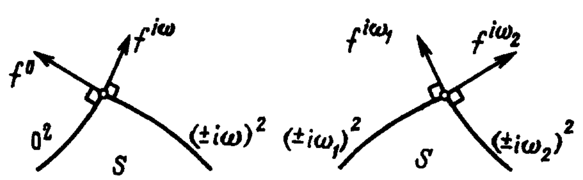

Fig. 1.

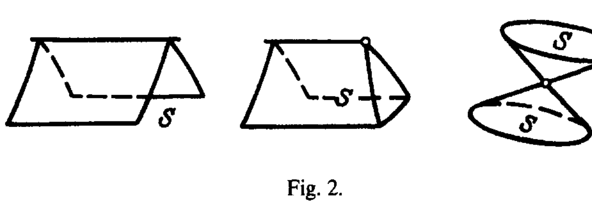

Fig. 2.

$$K_{(\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2} = \{\mathbf{e} : (\mathbf{f}^{i\omega_1}, \mathbf{e}) \leqslant 0, (\mathbf{f}^{i\omega_2}, \mathbf{e}) \leqslant 0, (\mathbf{f}^{i\omega_3}, \mathbf{e}) \leqslant 0\} \tag{2.4}$$

Three edges issue from the singular point $0^2(\pm i\omega_1)^2(\pm i\omega_2)^2$, in the directions $\mathbf{e}_{\tau 1}$, $\mathbf{e}_{\tau 2}$, $\mathbf{e}_{\tau 3}$, each orthogonal to two of the vectors $\mathbf{f}^0$, $\mathbf{f}^{i\omega_1}$, $\mathbf{f}^{i\omega_2}$ and pointing in the direction opposite to that of the third vector. For example, $\mathbf{e}_{\tau 1} = \mathbf{f}^{i\omega_1} \times \mathbf{f}^{i\omega_2}$, where the sign is chosen so that $(\mathbf{f}^0, \mathbf{e}_{\tau 1}) < 0$. Similar relations hold for the direction vectors of the edges in the case of the singularity $(\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2$.

The point 00 corresponds to a singularity of the "cone" type (Fig. 2). The bifurcation of the double eigenvalue $\lambda_0 = 0$ in this case is described by Eq. (1.8). The bifurcation takes place along the imaginary axis (stability) if $(\mathbf{k}_{12}, \mathbf{e}) < (\mathbf{k}_{1,} {}_{\mathbf{e}})((\mathbf{k}_{2,} {}_{\mathbf{e}})$ and along the real axis (instability) if the reverse inequality is true. The tangent cone to the stability domain at a point of type 00 has the form

$$K_{00} = \{\mathbf{e} : (\mathbf{k}_{12}, \mathbf{e})^2 \leqslant (\mathbf{k}_1, \mathbf{e})(\mathbf{k}_2, \mathbf{e})\} \tag{2.5}$$

A point of the type $(\pm i\omega)(\pm i\omega)$ in the case of a positive product $b_1 b_2$, where b_1 and b_2 are defined in (1.5), is an interior point of the stability domain and does not form a singularity. This follows from the fact that the condition for bifurcation of a double eigenvalue $\lambda = i\omega$ along the imaginary axis (1.6) takes the form $-(\mathbf{g}_1, \mathbf{e})^2 - (\mathbf{g}_2, \mathbf{e})^2 - (\mathbf{g}_3, \mathbf{e})^2 < 0$ and is satisfied for all directions $\mathbf{e}$ (if the vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are linearly independent). If $b_1 b_2 < 0$, one obtains at $(\pm i\omega)(\pm i\omega)$ a "cone" type singularity (Fig. 2). The condition for bifurcation along the imaginary axis in this case is $(\mathbf{g}_1, \mathbf{e})^2 \geqslant (\mathbf{g}_2, \mathbf{e})^2 + (\mathbf{g}_3, \mathbf{e})^2$, whence we obtain the following expression for the tangent cone to the stability domain

$$K_{(\pm i\omega)(\pm i\omega)} = \{\mathbf{e} : (\mathbf{g}_1, \mathbf{e})^2 \geqslant (\mathbf{g}_2, \mathbf{e})^2 + (\mathbf{g}_3, \mathbf{e})^2\} \tag{2.6}$$

Define vectors

$$\mathbf{a} = \mathbf{g}_2 \times \mathbf{g}_3, \quad \mathbf{b} = -\mathbf{g}_2 \times \mathbf{g}_1, \quad \mathbf{c} = -\mathbf{g}_1 \times \mathbf{g}_3$$

$$\mathbf{a}' = \mathbf{k}_{12} \times (\mathbf{k}_2 - \mathbf{k}_1), \quad \mathbf{b}' = -\mathbf{k}_{12} \times (\mathbf{k}_1 + \mathbf{k}_2), \quad \mathbf{c}' = \mathbf{k}_2 \times \mathbf{k}_1$$

In terms of these vectors the tangent cone (2.6) may be written in the form

$$K_{(\pm i\omega)(\pm i\omega)} = \{\mathbf{e} : \mathbf{e} = t(\mathbf{a} + d(\mathbf{b} \sin\alpha + \mathbf{c}\cos\alpha)), \quad t, \alpha \in \mathbb{R}, \quad d \in [0, 1]\} \tag{2.7}$$

The tangent cone (2.6) admits of an analogous representation, with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ replaced by $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$.

The vectors just defined have a readily understood geometric meaning. For example, the vector $t(\mathbf{a} + \mathbf{b}\sin\alpha + \mathbf{c}\cos\alpha)$, with t fixed and α varied from 0 to π, describes an ellipse in the parameter space. This ellipse is section of the cone by a plane parallel to the vectors $\mathbf{b}$ and $\mathbf{c}$ (Fig. 3).

Singular points of the type 0^4, $(\pm i\omega)^3$, $0^4(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$, $(\pm i\omega)^4$, 0^6 are associated with degenerate tangent cones and will be considered in the next section.

In the case of general position, the vectors defining the tangent cone (2.1)–(2.7) form linearly independent systems. The linear independence of the vectors defining the tangent cones may be used as a criterion for a given singularity to be in general position. Note that in order to evaluate a tangent cone to the stability domain one needs to know only the eigenvectors and generalized eigenvectors for multiple eigenvalues, as well as the first derivatives of the matrix $\mathbf{A}$ with respect to the parameters, evaluated at the relevant singular point of the boundary.

3. SINGULARITIES WITH DEGENERATE TANGENT CONES

In this section we investigate singularities of the BSD in general position to which there correspond degenerate tangent cones. For two-parameter families of Hamiltonian matrices such singularities are "cuspidal points" 0^4 and $(\pm i\omega)^3$ (Fig. 4) [1]. In the case of three parameters these are "cuspidal edges" 0^4 and $(\pm i\omega)^3$, "truncated cuspidal edges" 0^4 and $(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$ and "trihedral spires" 0^6, $(\pm i\omega)^4$ (Fig. 5). The BSD for a "trihedral spire" singularity consists of three edges which converge at the singularity point, along the tangent, to a common ray. Two of these edges are of the "cuspidal edge" type and the third is of the "trihedral angle" type.

Expressions for the tangent cones at these singular points are given in the following theorem.

Theorem. The tangent cones to the stability domain at singular points of the boundary of types 0^4, $(\pm i\omega)^3$, 0^6, $(\pm i\omega)^4$, $0^4(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$ are degenerate and have the following forms

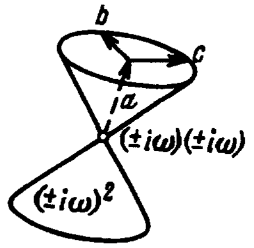

Fig. 3.

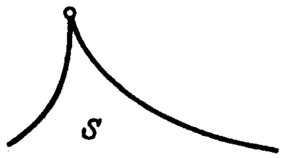

Fig. 4.

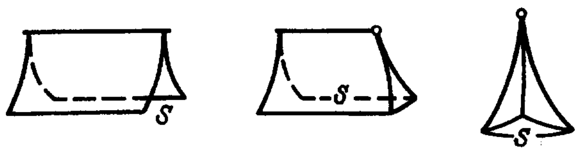

Fig. 5.

$$K_{0^4} = \{\mathbf{e} : (\mathbf{f}_0^0, \mathbf{e}) = 0, (\mathbf{f}_2^0, \mathbf{e}) \leqslant 0\} \tag{3.1}$$

$$K_{(\pm i\omega)^3} = \{\mathbf{e} : (\mathbf{f}_0^{i\omega}, \mathbf{e}) = 0, (\mathbf{f}_1^{i\omega}, \mathbf{e}) \leqslant 0\} \tag{3.2}$$

$$K_{0^6} = \{\mathbf{e} : (\mathbf{f}_0^0, \mathbf{e}) = 0, (\mathbf{f}_2^0, \mathbf{e}) = 0, (\mathbf{f}_4^0, \mathbf{e}) \geqslant 0\} \tag{3.3}$$

$$K_{(\pm i\omega)^4} = \{\mathbf{e} : (\mathbf{f}_0^{i\omega}, \mathbf{e}) = 0, (\mathbf{f}_1^{i\omega}, \mathbf{e}) = 0, (\mathbf{f}_2^{i\omega}, \mathbf{e}) \leqslant 0\} \tag{3.4}$$

$$K_{0^4(\pm i\omega)^2} = \{\mathbf{e} : (\mathbf{f}_0^0, \mathbf{e}) = 0, (\mathbf{f}_2^0, \mathbf{e}) \leqslant 0, (\mathbf{f}^{i\omega}, \mathbf{e}) \leqslant 0\} \tag{3.5}$$

$$K_{(\pm i\omega)^3 0^2} = \{\mathbf{e} : (\mathbf{f}_0^{i\omega}, \mathbf{e}) = 0, (\mathbf{f}_1^{i\omega}, \mathbf{e}) \leqslant 0, (\mathbf{f}^0, \mathbf{e}) \leqslant 0\} \tag{3.6}$$

$$K_{(\pm i\omega_1)^3(\pm i\omega_2)^2} = \{\mathbf{e} : (\mathbf{f}_0^{i\omega_1}, \mathbf{e}) = 0, (\mathbf{f}_1^{i\omega_1}, \mathbf{e}) \leqslant 0, (\mathbf{f}^{i\omega_2}, \mathbf{e}) \leqslant 0\} \tag{3.7}$$

where the superscript indicates the eigenvalues at which the vector in question is evaluated. Formulae (3.1)–(3.7) hold if the vectors defining the tangent cones constitute a linearly independent system. In general position this condition is satisfied.

Let us describe the main steps of the proof.†

It has been shown [1] that any family of Hamiltonian matrices $\mathbf{JA}(\mathbf{h})$ with a matrix $\mathbf{JA}(\mathbf{h}_0) = \mathbf{JA}_0$ may be represented in the form

$$\mathbf{JA}(\mathbf{h}) = \mathbf{C}(\mathbf{h})\mathbf{JA}'(\mathbf{g}(\mathbf{h}))\mathbf{C}^{-1}(\mathbf{h})$$

where $\mathbf{C}(\mathbf{h})$ is a family of non-singular matrices, $\mathbf{g} = \mathbf{g}(\mathbf{h})$, $\mathbf{g}(\mathbf{h}_0) = 0$, $\mathbf{g} \in \mathbb{R}^d$ is a smooth mapping of a neighbourhood of $\mathbf{h} = \mathbf{h}_0$ into a neighbourhood of $\mathbf{g} = 0$. The family $\mathbf{JA}'(\mathbf{g})$ is called a versal deformation and is chosen in accordance with the Williamson normal form of the matrix $\mathbf{A}_0$ and $\mathbf{g}$ is the parameter vector of a miniversal deformation [1]. To describe the family $\mathbf{JA}'(\mathbf{g})$ it is sufficient to indicate the form of the Hamiltonian $H'(\mathbf{g}) = \mathbf{x}^T\mathbf{A}'(\mathbf{g})\mathbf{x}/2$. The Hamiltonian $H'(\mathbf{g})$ may be expressed as a sum [1]

$$H'(\mathbf{g}) = H'_{(1)}(\mathbf{g}^{(1)}) + H'_{(2)}(\mathbf{g}^{(2)}) + \dots \tag{3.8}$$

where $H'_{(j)}(\mathbf{g}^{(j)} = \mathbf{x}^{(j)T}\mathbf{A}'_j(\mathbf{g}^{(j)})\mathbf{x}^{(j)}/2$ ($j = 1, 2, \dots$) are the Hamiltonians corresponding to the different eigenvalues and which depend on the corresponding vectors of variables of the Hamiltonian $\mathbf{x}^{(j)}$ and parameter vector $\mathbf{g}^{(j)}$. The vectors $\mathbf{x}^{(j)}$ and $\mathbf{g}^{(j)}$, evaluated for all $j = 1, 2, \dots$, and taken together, form

†A detailed proof may be found in A. A. MAILYBAEV and A. P. SEIRANYAN., The stability domains of linear Hamiltonian systems. Preprint No. 37–98. Institute of Mechanics, Moscow State University, Moscow 1998.

vectors $\mathbf{x}$ and $\mathbf{g}$, respectively, and the matrix $\mathbf{A}'(0)$ corresponding to the Hamiltonian $H'(0)$ of (3.8) is the Williamson normal form of $\mathbf{A}_0$.

Since the matrices $\mathbf{JA}(\mathbf{h})$ and $\mathbf{JA}'(\mathbf{g}(\mathbf{h}))$ have the same eigenvalues, the family $\mathbf{JA}(\mathbf{h})$ is stable if and only if the family $\mathbf{JA}'(\mathbf{g}(\mathbf{h}))$ is stable. Thus, when the mapping $\mathbf{g}(\mathbf{h})$ is known, we can investigate the stability of the family $\mathbf{JA}'(\mathbf{g})$ instead of the stability of the family $\mathbf{JA}(\mathbf{h})$.

Representation (3.8) enables us to split the system of Hamilton equations for $H'(g)$ into independent systems for each term $H'_{(j)}(\mathbf{g}^{(j)})$. The stability of the family $\mathbf{JA}'(\mathbf{g})$ in the neighbourhood of $\mathbf{g} = 0$ will then be determined by the stability of the systems associated with multiple eigenvalues of the matrix $\mathbf{JA}_0$, and with those eigenvalues only. The Hamiltonians $H'_{(j)}(\mathbf{g}^{(j)})$ have been derived [1] in terms of the Jordan structure of the corresponding eigenvalues. Using these representations, one can find the tangent cones to the stability domain of the family $\mathbf{JA}'(\mathbf{g})$ for the types of singularity under consideration. For a multiple pure imaginary eigenvalue λ_0 with one Jordan cell of order k, one can prove the following relation for the direction vectors $\mathbf{e}$ and $\mathbf{e}'$ in the tangent cones in the spaces of the parameters $\mathbf{h} \in \mathbb{R}^n$ and $\mathbf{g} \in \mathbb{R}^d$

$$(\mathbf{f}_j, \mathbf{e}) = (\mathbf{f}'_j, \mathbf{e}'), \quad j = 0, \ldots, k-1 \tag{3.9}$$

where $\mathbf{f}_j$, $\mathbf{f}'_j$ are the vectors evaluated by formulae (1.2) for the eigenvalue λ_0 of the families $\mathbf{JA}(\mathbf{h})$ and $\mathbf{JA}'(\mathbf{g})$ at points $\mathbf{h} = \mathbf{h}_0$ and $\mathbf{g} = 0$, respectively. Equations (3.9) enable us using the expressions for the tangent cones to the stability domain of the family $\mathbf{JA}'(\mathbf{g})$, to find the tangent cones to the stability domain of the family $\mathbf{JA}(\mathbf{h})$ in the original parameter space. One thus obtains the expressions for the tangent cones stated in the theorem.

4. SINGULARITIES OF BOUNDARIES OF STABILITY DOMAINS OF GYROSCOPIC SYSTEMS

Consider a linear autonomous gyroscopic system

$$\mathbf{M}\ddot{\Phi} + \mathbf{G}\dot{\Phi} + \mathbf{K}\Phi = 0 \tag{4.1}$$

where $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_m)^T \in \mathbb{R}^m$ is the vector of generalized coordinates; $\mathbf{M}$, $\mathbf{G}$ and $\mathbf{K}$ are real matrices of order $m \times m$ and smooth functions of the parameter vector $\mathbf{h} \in \mathbb{R}^n$, such that $\mathbf{M}^T = \mathbf{M} > 0$, $\mathbf{G}^T = -\mathbf{G}$, $\mathbf{K}^T = \mathbf{K}$. Writing the solution of Eq. (4.1) in the form $\Phi = \mathbf{U}e^{\lambda t}$, we arrive at the eigenvalue problem

$$[\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}]\mathbf{U} = 0$$

Equation (4.1) may be written in the form of Eq. (1.1) [5], where

$$\mathbf{A} = \begin{Vmatrix} \mathbf{K} - \mathbf{G}\mathbf{M}^{-1}\mathbf{G}/4 & \mathbf{G}\mathbf{M}^{-1}/2 \\ -\mathbf{M}^{-1}\mathbf{G}/2 & \mathbf{M}^{-1} \end{Vmatrix}, \quad \mathbf{x} = \begin{Vmatrix} \Phi \\ \Psi \end{Vmatrix}, \quad \Psi = \mathbf{M}\dot{\Phi} + \mathbf{G}\Phi/2 \tag{4.2}$$

The matrix $\mathbf{A}$ is symmetric and depends smoothly on the vector $\mathbf{h}$.

The systems of equations (4.1) and (1.1), (4.2), are the equations of motion written in Lagrange and Hamilton forms, respectively. The stability domains of the trivial solutions of systems (4.1) and (1.1), (4.2) coincide. The singularities of the BSD in general position for Hamiltonian and gyroscopic systems which depend on parameters are identical. All results obtained in Sections 2 and 3 for singularities of the BSD of Hamiltonian systems are also true for gyroscopic systems (4.1), provided the family of matrices $\mathbf{A}(\mathbf{h})$ is taken in the form of (4.2). Substituting (4.2) into the relations of Section 1, one can write the expressions for the vectors defining the tangent cones directly in terms of the gyroscopic system (4.1).

1. Suppose that when $\mathbf{h} = \mathbf{h}_0$ the gyroscopic system has an eigenvalue $\lambda = i\omega$ (or $\lambda = 0$) of multiplicity k, associated with which is a Jordan chain of vectors $\mathbf{U}_0, \ldots, \mathbf{U}_{k-1}$ satisfying the equations

$$\begin{aligned} &\mathbf{Q}\mathbf{U}_0 = 0 \\ &\mathbf{Q}\mathbf{U}_1 + \mathbf{Q}_\lambda\mathbf{U}_0 = 0 \\ &\mathbf{Q}\mathbf{U}_2 + \mathbf{Q}_\lambda\mathbf{U}_1 + \mathbf{Q}_{\lambda\lambda}\mathbf{U}_0/2 = 0 \\ &\ldots \\ &\mathbf{Q}\mathbf{U}_{k-1} + \mathbf{Q}_\lambda\mathbf{U}_{k-2} + \mathbf{Q}_{\lambda\lambda}\mathbf{U}_{k-3}/2 = 0 \end{aligned} \tag{4.3}$$

where we have used the following notation for the matrix operators

$$\mathbf{Q} = \lambda^2\mathbf{M}(\mathbf{h}_0) + \lambda\mathbf{G}(\mathbf{h}_0) + \mathbf{K}(\mathbf{h}_0),\quad \mathbf{Q}_\lambda = 2\lambda\mathbf{M}(\mathbf{h}_0) + \mathbf{G}(\mathbf{h}_0),\quad \mathbf{Q}_{\lambda\lambda} = 2\mathbf{M}(\mathbf{h}_0)$$

The chain of vectors $\mathbf{V}_0, \ldots, \mathbf{V}_{k-1}$ for the adjoint problem satisfies Eqs (4.3) with the operators $\mathbf{Q}$, $\mathbf{Q}_\lambda$, $\mathbf{Q}_{\lambda\lambda}$ replaced by their adjoints $\mathbf{Q}^*$, $\mathbf{Q}^*_\lambda$, $\mathbf{Q}^*_{\lambda\lambda}$.

The normalization conditions for the vectors $\mathbf{U}_j$ and $\mathbf{V}_j$ are

$$\mathbf{V}_0^*\mathbf{Q}_\lambda\mathbf{U}_{k-1} + \tfrac{1}{2}\mathbf{V}_0^*\mathbf{Q}_{\lambda\lambda}\mathbf{U}_{k-2} = 1 \tag{4.4}$$

$$\tfrac{1}{2}\mathbf{V}_{j-1}^*\mathbf{Q}_{\lambda\lambda}\mathbf{U}_{k-1} + \mathbf{V}_j^*\mathbf{Q}_\lambda\mathbf{U}_{k-1} + \tfrac{1}{2}\mathbf{V}_j^*\mathbf{Q}_{\lambda\lambda}\mathbf{U}_{k-2} = 0,\quad j = 1,\ldots,k-1$$

Then the components of the vectors $\mathbf{f}_j = (f_j^1, \ldots, f_j^n)^T \in \mathbb{R}^n$ $(j = 0, \ldots, k-1)$ are defined by the relations

$$f_j^l = i^{k-j}\left(\sum_{r=0}^{j}\mathbf{V}_r^*\mathbf{Q}_l\mathbf{U}_{j-r} + \sum_{r=0}^{j-1}\mathbf{V}_r^*\mathbf{Q}_{\lambda l}\mathbf{U}_{j-r-1} + \frac{1}{2}\sum_{r=0}^{j-2}\mathbf{V}_r^*\mathbf{Q}_{\lambda\lambda l}\mathbf{U}_{j-r-2}\right) \tag{4.5}$$

$$Q_l = \lambda^2\frac{\partial\mathbf{M}}{\partial h_l} + \lambda\frac{\partial\mathbf{G}}{\partial h_l} + \frac{\partial\mathbf{K}}{\partial h_l},\quad \mathbf{Q}_{\lambda l} = 2\lambda\frac{\partial\mathbf{M}}{\partial h_l} + \frac{\partial\mathbf{G}}{\partial h_l},\quad \mathbf{Q}_{\lambda\lambda l} = 2\frac{\partial\mathbf{M}}{\partial h_l}$$

The derivatives with respect to the parameters are evaluated at the point $\mathbf{h} = \mathbf{h}_0$.

When $k = 2$, the components of $\mathbf{f}_0$ (or simply $\mathbf{f}$) may be written in the form

$$f^l = -\frac{\mathbf{U}_0^*\mathbf{Q}_l\mathbf{U}_0}{\mathbf{U}_1^*\mathbf{Q}\mathbf{U}_1 + \mathbf{U}_0^*\mathbf{Q}_{\lambda\lambda}\mathbf{U}_0/2} \tag{4.6}$$

2. Consider a double eigenvalue $\lambda = i\omega \neq 0$, associated with which are two eigenvectors $\mathbf{U}'$ and $\mathbf{U}''$. We introduce an orthogonality condition $\mathbf{U}'^*\mathbf{Q}_\lambda\mathbf{U}'' = 0$. Then the constants b_1, b_2 and the components of the vectors $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \in \mathbb{R}^n$ are given by the relations

$$ib_1 = \mathbf{U}'^*\mathbf{Q}_\lambda\mathbf{U}',\quad ib_2 = \mathbf{U}''^*\mathbf{Q}_\lambda\mathbf{U}'' \tag{4.7}$$

$$g_1^l = b_2\mathbf{U}'^*\mathbf{Q}_l\mathbf{U}' + b_1\mathbf{U}''^*\mathbf{Q}_l\mathbf{U}'',\quad g_2^l + ig_3^l = 2\sqrt{|b_1b_2|}\,\mathbf{U}'^*\mathbf{Q}_l\mathbf{U}''$$

3. Consider a double eigenvalue $\lambda = 0$ with two eigenvectors $\mathbf{U}'$ and $\mathbf{U}''$, which we choose to be real and such that the normalization conditions $\mathbf{U}'^T\mathbf{G}_0\mathbf{U}'' = 1$, $\mathbf{G}_0 = \mathbf{G}(\mathbf{h}_0)$ are satisfied. The expressions for the components of the vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \in \mathbb{R}^n$ become

$$k_1^l = \mathbf{U}'^T\frac{\partial\mathbf{K}}{\partial p_l}\mathbf{U}',\quad k_2^l = \mathbf{U}''^T\frac{\partial\mathbf{K}}{\partial p_l}\mathbf{U}'',\quad k_{12}^l = \mathbf{U}'^T\frac{\partial\mathbf{K}}{\partial p_l}\mathbf{U}'',\quad l = 1,\ldots,n \tag{4.8}$$

The derivatives with respect to the parameters are evaluated at the point $\mathbf{h} = \mathbf{h}_0$.

Note that the matrices of a Hamiltonian or gyroscopic system and their first derivatives with respect to the parameters, evaluated at a singular point of the boundary, enable one to determine the geometry of the stability domain in a neighbourhood of that point, in the first approximation.

5. APPROXIMATE INVESTIGATION OF THE STABILITY OF THE OSCILLATIONS OF A TUBE CONTAINING A FLOWING FLUID

As an example of a two-parameter gyroscopic system, we will consider an elastic tube on a hinged support, through which a fluid is flowing. The linear differential equation for the vibrations of the tube and the boundary conditions have the following form [9]

$$(m+m_f)\frac{\partial^2 w}{\partial t^2} + 2v_f m_f\frac{\partial^2 w}{\partial x\partial t} + EJ\frac{\partial^4 w}{\partial x^4} + m_f v_f^2\frac{\partial^2 w}{\partial x^2} = 0 \tag{5.1}$$

$$w(0) = \left(EJ\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = 0,\quad w(l) = \left(EJ\frac{\partial^2 w}{\partial x^2}\right)_{x=l} = 0$$

where $w(x, t)$ is the deflection of the tube, m, EJ and l are the mass per unit length, flexural stiffness and length of the tube, m_f and v_f are the mass per unit length and the flow velocity of the fluid and t denotes the time. The terms shown in Eq. (5.1) describe the inertial, Coriolis, elastic and centrifugal forces acting on the tube. Damping is ignored.

Solving Eq. (5.1) by the Bubnov–Galerkin method and retaining two terms in the expansions

$$w(x,t) = \varphi_1(t)\sin\frac{\pi x}{l} + \varphi_2(t)\sin\frac{2\pi x}{l}$$

we obtain a system of ordinary differential equations (4.1) for the functions $\varphi_1(t)$ and $\varphi_2(t)$, where [9]

$$\mathbf{M} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{G} = \sqrt{\alpha\Lambda}\begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \quad \mathbf{K} = \begin{Vmatrix} 1-\Lambda & 0 \\ 0 & 16-4\Lambda \end{Vmatrix} \tag{5.2}$$

We have introduced here dimensionless parameters α and Λ characterizing the relative mass and flow velocity [9]

$$\alpha = \left(\frac{16}{3\pi}\right)^2 \frac{m_f}{m+m_f}, \quad \Lambda = \frac{m_f v_f^2 l^2}{\pi^2 \mathrm{EJ}} \tag{5.3}$$

If the relative mass varies in the range $0 \leqslant m_f/m < \infty$, then α varies in the range $0 \leqslant \alpha \leqslant (16/3\pi)^2 \approx 2.882$. The flow velocity parameter Λ is non-negative, $0 \leqslant \Lambda < \infty$.

The characteristic equation for the system, in view of (5.2), is [9]

$$\lambda^4 + \lambda^2(17 - 5\Lambda + \alpha\Lambda) + 4(1-\Lambda)(4-\Lambda) = 0 \tag{5.4}$$

The point (4, 3/4) is a singular point of the BSD of the type 0^4. Indeed, at this point both coefficients of polynomial (5.4) vanish. Therefore $\lambda = 0$ is a four-fold root of Eq. (5.4). Setting

$$\mathbf{Q} = \begin{Vmatrix} -3 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \mathbf{Q}_\lambda = \begin{Vmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{Vmatrix}, \quad \mathbf{Q}_{\lambda\lambda} = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix} \tag{5.5}$$

in Eqs (4.3), we find the corresponding chains of four right and left eigenvalues and associated eigenvalues, satisfying the normalization conditions (4.4)

$$\mathbf{U}_0 = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad \mathbf{U}_1 = \begin{Vmatrix} -\sqrt{3}/3 \\ 0 \end{Vmatrix}, \quad \mathbf{U}_2 = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad \mathbf{U}_3 = \begin{Vmatrix} -4\sqrt{3}/9 \\ 0 \end{Vmatrix} \tag{5.6}$$

$$\mathbf{V}_0 = \begin{Vmatrix} 0 \\ -3 \end{Vmatrix}, \quad \mathbf{V}_1 = \begin{Vmatrix} -\sqrt{3} \\ 0 \end{Vmatrix}, \quad \mathbf{V}_2 = \begin{Vmatrix} 0 \\ 4 \end{Vmatrix}, \quad \mathbf{V}_3 = \begin{Vmatrix} \sqrt{3} \\ 0 \end{Vmatrix}$$

The presence of linearly dependent vectors should not confuse us, as the chain (4.3) establishes a relationship among the three vectors $\mathbf{U}_j$, $\mathbf{U}_{j-1}$, $\mathbf{U}_{j-2}$. A similar relationship exists for the triple of vectors $\mathbf{V}_j$, $\mathbf{V}_{j-1}$, $\mathbf{V}_{j-2}$.

Using (5.2), (5.6) and (4.5), we evaluate the vectors $\mathbf{f}_0$ and $\mathbf{f}_2$ defining the tangent cones to the stability domain at the point 0^4 ($\Lambda = 4$, $\alpha = 3/4$)

$$\mathbf{f}_0 = (12, 0)^T, \quad \mathbf{f}_2 = (17/4, -4)^T \tag{5.7}$$

Thus, formulae (3.1) and (5.7) yield the tangent cone to the stability domain

$$K_{0^4} = \left\{\mathbf{e} = (e_1, e_2)^T : \; e_1 = 0, \; e_2 \geqslant 0\right\} \tag{5.8}$$

Tangent cone (5.8) consists of a single direction, which defines the orientation of a "cuspidal point" singularity in the space of the parameters Λ, α.

The stability domain of the system may be found by analytical means. A necessary condition for stability is that the coefficients of the biquadratic equation (5.4) and its discriminant should be non-negative. The corresponding strict inequalities define the stability domains in the plane of the parameters $\mathbf{h} = (\Lambda, \alpha)^T$

$$1)\ 0 < \Lambda < 1; \quad 2)\ \Lambda > 4, \quad \alpha > 5 - \frac{17}{\Lambda} + 4\left[\left(\frac{1}{\Lambda} - 1\right)\left(\frac{4}{\Lambda} - 1\right)\right]^{1/2}$$

as shown in Fig. 6. Domain 2 is the domain of gyroscopic stability. This result agrees with (5.8), since both the curves making up the BSD are tangent at the point 0^4 to the ray $\Lambda = 0$, $\alpha \geqslant 0$ (Fig. 6).

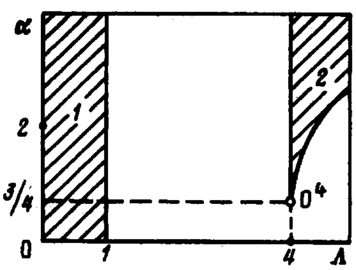

Fig. 6.

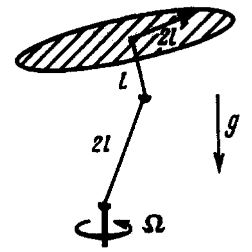

Fig. 7.

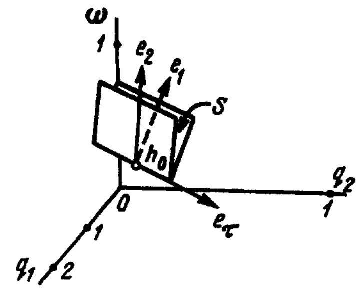

Fig. 8.

6. THE STABILITY OF THE ROTATION OF A STATICALLY UNSTABLE MECHANICAL SYSTEM

Consider a mechanical system in a gravity field. The system consists of a vertically mounted motor, with two rigid, weightless rods of lengths $2l$ and l attached in sequence to the rotor of the motor by elastic ball-and-socket joints. The end of the second rod is rigidly attached to the centre of a flat disc of mass m and radius $2l$, in such a way that the rod is perpendicular to the plane of the disc (Fig. 7). As generalized coordinates we take the set of Krylov angles α_i, β_i, γ_i ($i = 1, 2$), which define the position of each rod in a frame of reference attached to the rotor. Such mechanical systems were considered previously in [10].

Consider the stability of the rotation of the system about the vertical axis $\alpha_i = \beta_i = \gamma_i = 0$ ($i = 1, 2$), at constant angular velocity. The system of linearized equations of motion splits into two systems, one of which contains only γ_1 and γ_2 and is always stable. The other system has the form (4.1), with $\Phi = (\alpha_1, \beta_1, \alpha_2, \beta_2)^T$, and the non-zero elements of the matrices $\mathbf{M} = || m_{ij} ||$, $\mathbf{G} = || g_{ij} ||$ and $\mathbf{K} = || k_{ij} ||$ are as follows:

$$m_{11} = m_{22} = -g_{14} = g_{41} = g_{23} = -g_{32} = 4, \quad -g_{12} = g_{21} = 8$$

$$m_{13} = m_{31} = m_{24} = m_{42} = m_{33} = m_{44} = -g_{34} = g_{43} = 2$$

$$k_{11} = k_{22} = (q_1 + q_2 - 2)/\omega^2 - 4, \quad k_{33} = k_{44} = (q_2 - 1)/\omega^2$$

$$k_{13} = k_{31} = k_{24} = k_{42} = -q_2/\omega^2 - 2$$

The dimensionless parameters q_1, q_2 and ω characterize the stiffnesses of the sockets and the angular velocity of rotation of the rotor.

Consider the point $\mathbf{h}_0 = (3/2, 2\sqrt{(2)} - 5/2, 2 - \sqrt{(2)})^T$ in the space of the parameters $\mathbf{h} = (q_1, q_2, \omega)^T$. The characteristic equation of the system at the point has two pairs of double pure imaginary roots $\lambda = \pm i(2 - \sqrt{2})/4$, $\lambda = \pm i(2 + \sqrt{(2)})/4$, associated with which are Jordan chairs of order two. Consequently, the stability domain in the space of the parameters $\mathbf{h} = (q_1, q_2, \omega)^T$ has a singularity of the "dihedral angle" type, $(\pm i\omega_1)^2(\pm i\omega_2)^2$. The vectors $\mathbf{f}^{i\omega_1}$ and $\mathbf{f}^{i\omega_2}$ defining the tangent cones (2.2) are as follows:

$$\mathbf{f}^{i\omega_1} = -\frac{10 + 7\sqrt{2}}{8} \left\| \begin{matrix} 1 \\ 3 + 2\sqrt{2} \\ 2 \end{matrix} \right\| \qquad \mathbf{f}^{i\omega_2} = -\frac{2 + \sqrt{2}}{8} \left\| \begin{matrix} 3 + 2\sqrt{2} \\ 9 + 4\sqrt{2} \\ 6 - 4\sqrt{2} \end{matrix} \right\|$$

Using a vector $\mathbf{e}_\tau$ tangent to the edge of the "dihedral angle" and vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ orthogonal to the edge and tangent to the faces of the "dihedral angle", having the form

$$\mathbf{e}_\tau = \mathbf{f}^{i\omega_1} \times \mathbf{f}^{i\omega_2}, \quad \mathbf{e}_1 = \mathbf{f}^{i\omega_1} \times \mathbf{e}_\tau, \quad \mathbf{e}_2 = \mathbf{e}_\tau \times \mathbf{f}^{i\omega_2}$$

we can write the tangent cones to the stability domain (2.2) in the form

$$K_{(\pm i\omega_1)^2(\pm i\omega_2)^2} = \{\mathbf{e} : \mathbf{e} = \alpha\mathbf{e}_\tau + \beta\mathbf{e}_1 + \gamma\mathbf{e}_2, \alpha \in \mathbb{R}, \ \beta, \gamma \geqslant 0\}$$

The vectors $\mathbf{e}_\tau$, $\mathbf{e}_1$ and $\mathbf{e}_2$ define the geometry of the stability domain in the neighbourhood of the singular point $\mathbf{h} = \mathbf{h}_0$ in the linear approximation (Fig. 8).

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